The complexity of determining the rainbow vertex-connection of graphs*

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Abstract

A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors, which was introduced by Krivelevich and Yuster. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. In this paper, we study the computational complexity of vertex-rainbow connection of graphs and prove that computing $rvc(G)$ is NP-Hard. Moreover, we show that it is already NP-Complete to decide whether $rvc(G) = 2$. We also prove that the following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.

Keywords: coloring; rainbow vertex-connection; computational complexity

AMS Subject Classification (2010): 05C15, 05C40, 68Q25, 68R10.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of Bondy and Murty [1].

An edge-colored graph is *rainbow connected* if any two vertices are connected by a path whose edges have distinct colors. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [4]. The *rainbow connection number* of a connected

*Supported by NSFC and "the Fundamental Research Funds for the Central Universities".

graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. Observe that $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of G. It is easy to verify that $rc(G)=1$ if and only if G is a complete graph, that $rc(G)=n-1$ if and only if G is a tree. There are some approaches to study the bounds of $rc(G)$, we refer to [2, 5, 7].

In [5], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. A vertex-colored graph is *rainbow vertex-connected* if any two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. An easy observation is that if G is of order n then $rvc(G) \leq n-2$ and $rvc(G)=0$ if and only if G is a complete graph. Notice that $rvc(G) \geq diam(G)-1$ with equality if the diameter is 1 or 2. For rainbow connection and rainbow vertex-connection, some examples are given to show that there is no upper bound for one of parameters in terms of the other in [5]. Krivelevich and Yuster [5] proved that if G is a graph with n vertices and minimum degree δ, then $rvc(G) < 11n/\delta$. In [6], the authors improved this bound for given order n and minimum degree δ.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connectivity can also find applications in networking. Suppose we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. The minimum number of used channels is exactly the rainbow connection of the underlying graph.

The computational complexity of rainbow connection has been studied. In [2], Caro et al. conjectured that computing $rc(G)$ is an NP-Hard problem, as well as that even deciding whether a graph has $rc(G)=2$ is NP-Complete. In [3], Chakraborty et al. confirmed this conjecture. Motivated by the proof of [3], we consider the computational complexity of rainbow vertex-connection $rvc(G)$ of graphs. It is not hard to image that this problem is also NP-hard, but a rigorous proof is necessary. This paper is to give such a proof, which follows a similar idea of [3], but by reducing 3-SAT problem to some other new problems, that computing $rvc(G)$ is NP-Hard. Moreover, we show that it is already NP-Complete to decide whether $rvc(G)=2$. We also prove that the following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.

2 Rainbow vertex-connection.

For two problems A and B, we write $A \preceq B$, if problem A is polynomially reducible to problem B. Now, we give our first theorem.

Theorem 1 *The following problem is NP-Complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.*

Now we define Problem 1 and Problem 2 as follows. We will prove Theorem 1 by reducing Problem 1 to Problem 2, and then 3-SAT problem to Problem 1.

Problem 1 $s-t$ rainbow vertex-connection.

Given: Vertex-colored graph G with two vertices $s,\ t$.
Decide: Whether there is a rainbow vertex-connected path between s and t?

Problem 2 Rainbow vertex-connection.

Given: Vertex-colored graph G.
Decide: Whether G is rainbow vertex-connected under the coloring?

Lemma 1 *Problem 1 $\preceq$ Problem 2.*

Proof. Given a vertex-colored graph G with two vertices s and t. We want to construct a new graph G' with a vertex coloring such that G' is rainbow vertex-connected if and only if there is a rainbow vertex-connected path from s to t in G.

Let $V = \{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be vertices of G, where $v_1 = s$ and $v_n = t$. We construct G' as follows. Set

$$V' = V \cup \{s', t', a, b\}$$

and

$$E' = E \cup \{s's, t't\} \cup \{av_i, bv_i : i \in [n]\}.$$

Let c be the vertex coloring of G, we define the vertex coloring c' of G' by $c'(v_i) = c(v_i)$ for $i \in \{2, 3, \ldots, n-1\}$, $c'(s) = c'(a) = c_1$, $c'(t) = c'(b) = c_2$, where c_1, c_2 are the two new colors.

Suppose c' makes G' rainbow vertex-connected. Since each path Q from s' to t' must go through s and t, Q can not contain a and b as $c'(s) = c'(a) = c_1$ and $c'(t) = c'(b) = c_2$. Therefore, any rainbow vertex-connected path from s' to t' must contain a rainbow vertex-connected path from s to t in G. Thus there is a rainbow vertex-connected path from s to t in G under the coloring c.

Now assume that there is a rainbow vertex-connected path from s to t in G under the coloring c. To prove that G' is rainbow vertex-connected. First, the rainbow vertex-connected path from s' to v_i can be formed by going through s and b, then to v_i for $i \in \{2, 3, \ldots, n\}$. The rainbow vertex-connected path from s' to t' can go through s and t and a rainbow vertex-connected path from s to t in G. The rainbow vertex-connected

path from t' to v_i can be formed by going through t and a, then to v_i for $i \in \{2, 3, \ldots, n\}$. For the other pairs of vertices, there is a path between them with length less than 3, thus they are obvious rainbow vertex-connected. ∎

Lemma 2 *3-SAT* $\preceq$ *Problem 1.*

Proof. Let ϕ be an instance of 3-SAT with clauses $c_1, c_2, \ldots, c_m$ and variables $x_1, x_2, \ldots, x_n$. We construct a graph G_ϕ with special vertices s and t.

First, we introduce k new vertices $v_1^j, v_2^j, \ldots, v_k^j$ for each $x_j \in c_i$ and ℓ new vertices $\overline{v}_1^j, \overline{v}_2^j, \ldots, \overline{v}_\ell^j$ for each $\overline{x}_j \in c_i$. Without loss of generality, we assume that $k \geq 1$ and $\ell \geq 1$, otherwise ϕ can be simplified.

Next, for each v_a^j, $a \in [k]$, we introduce ℓ new vertices $v_{a1}^j, v_{a2}^j, \ldots, v_{a\ell}^j$, which form a path in this order. Similarly, for each $\overline{v}_b^j$, $b \in [\ell]$, we introduce k new vertices $\overline{v}_{1b}^j, \overline{v}_{2b}^j, \ldots, \overline{v}_{kb}^j$, which form a path in that order. Therefore, for $x_j \in c_i$, there are k paths of length $\ell - 1$, and for $\overline{x}_j \in c_i$, there are ℓ paths of length $k - 1$. For each path Q in c_i ($i \in [m]$), we join the original vertex of Q to the terminal vertices of all paths in c_{i-1}, where c_0 is the vertex s. And for each path in c_m, we join its terminal vertex to t. Thus, a new graph G_ϕ is obtained.

Now we define a vertex coloring of G_ϕ. For every variable x_j, we introduce $k \times \ell$ distinct colors $\alpha_{1,1}^j, \alpha_{1,2}^j, \ldots, \alpha_{k,\ell}^j$. We color vertices $v_{a1}^j, v_{a2}^j, \ldots, v_{a\ell}^j$ with colors $\alpha_{a,1}^j, \alpha_{a,2}^j, \ldots, \alpha_{a,\ell}^j$, respectively, and color $\overline{v}_{1b}^j, \overline{v}_{2b}^j, \ldots, \overline{v}_{kb}^j$ with colors $\alpha_{1,b}^j, \alpha_{2,b}^j, \ldots, \alpha_{k,b}^j$, respectively, where $a \in [k]$ and $b \in [\ell]$.

If G_ϕ contains a rainbow vertex-connected $s - t$ path Q, then Q must contain one of the newly built paths in each c_i, $i \in [m]$, and the path $v_{a1}^j v_{a2}^j \ldots v_{a\ell}^j$ and $\overline{v}_{1b}^j \overline{v}_{2b}^j \ldots \overline{v}_{kb}^j$ can not both appear in Q. If $v_{a1}^j v_{a2}^j \ldots v_{a\ell}^j$ appears in Q, set $x_j = 1$, and if $\overline{v}_{1b}^j \overline{v}_{2b}^j \ldots \overline{v}_{kb}^j$ appears in Q, set $x_j = 0$. Clearly, we have $\phi = 1$ in this assignment. ∎

3 Rainbow vertex-connection 2.

Before proceeding, we first define three problems.

Problem 3 Rainbow vertex-connection 2.

Given: Graph $G = (V, E)$.
Decide: Whether there is a vertex coloring of G with two colors such that all pairs $(u, v) \in V(G) \times V(G)$ are rainbow vertex-connected?

Problem 4 Subset rainbow vertex-connection 2.

Given: Graph $G = (V, E)$ and a set of pairs $P \subseteq V(G) \times V(G)$.
Decide: Whether there is a vertex coloring of G with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected?

Problem 5 Different subsets rainbow vertex-connection 2.

Given: Graph $G = (V, E)$ and two disjoint subsets V_1, V_2 of V with a one to one corresponding $f : V_1 \to V_2$.
Decide: Whether there is a vertex coloring of G with two colors such that G is rainbow vertex-connected and for each $v \in V_1$, v and $f(v)$ are assigned different colors.

In the following, we will reduce Problem 4 to Problem 3 and then reduce Problem 5 to Problem 4. Finally, we will show it is NP-Complete to decide whether $rvc(G) = 2$ by reducing 3-SAT problem to Problem 3.

Lemma 3 *Problem 4 $\preceq$ Problem 3.*

Proof. Given a graph $G = (V, E)$ and a set of pairs $P \subseteq V(G) \times V(G)$, we construct a graph $G' = (V', E')$ as follows.

For each vertex $v \in V$, we introduce a new vertex x_v; for every pair $(u, v) \in (V \times V) \setminus P$, we introduce two new vertices $x^1_{(u,v)}$ and $x^2_{(u,v)}$; we also add two new vertices s, t. Set

$$V' = V \cup \{x_v : v \in V\} \cup \{x^1_{(u,v)}, x^2_{(u,v)} : (u, v) \in (V \times V) \setminus P\} \cup \{s, t\}$$

and
$E' = E \cup \{vx_v : v \in V\} \cup \{ux^1_{(u,v)}, x^1_{(u,v)}x^2_{(u,v)}, x^2_{(u,v)}v : (u, v) \in (V \times V) \setminus P\} \cup \{sx^1_{(u,v)}, tx^2_{(u,v)} : (u, v) \in (V \times V) \setminus P\} \cup \{sx_v, tx_v : v \in V\}$.

Observe that G is a subgraph of G'. In the following, we will prove that G' is 2-rainbow vertex-connected if and only if there is a vertex coloring of G with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected.

Now suppose there is a vertex coloring of G' with two colors which makes G' rainbow vertex-connected. For each pair $(u, v) \in P$, the paths of length no more than 3 that connects u and v have to be in G. Thus, with the coloring all pairs in P are rainbow vertex-connected. On the other hand, let $c : V \to \{1, 2\}$ be one coloring of G such that all pairs $(u, v) \in P$ are rainbow vertex-connected. We extend the coloring as follows: $c(x_v) = 1$ for all $v \in P$, $c(x^1_{(u,v)}) = 1$ and $c(x^2_{(u,v)}) = 2$ for all $(u, v) \in (V \times V) \setminus P$, $c(s) = c(t) = 2$. We can see that G' is indeed rainbow vertex-connected under this coloring. ∎

Lemma 4 *Problem 5 $\preceq$ Problem 4.*

Proof. Given a graph $G = (V, E)$ and two disjoint subsets V_1, V_2 of V with a one to one corresponding f. Assume that $V_1 = \{v_1, v_2, \ldots, v_k\}$ and $V_2 = \{w_1, w_2, \ldots, w_k\}$ satisfying that $w_i = f(v_i)$ for each $i \in [k]$. We construct a new graph $G' = (V', E')$ as follows.

We introduce six new vertices $x^1_{v_iw_i}, x^2_{v_iw_i}, x^3_{v_iw_i}, x^4_{v_iw_i}, x^5_{v_iw_i}, x^6_{v_iw_i}$ for each pair (v_i, w_i), $i \in [n]$. We add a new vertex s. Set

$$V' = V \cup \{x^j_{v_iw_i} : i \in [k], j \in [6]\} \cup \{s\},$$

and
$E' = E \cup \{sx^5_{v_iw_i}, x^5_{v_iw_i}v_i, v_ix^1_{v_iw_i}, x^1_{v_iw_i}x^2_{v_iw_i}, x^2_{v_iw_i}x^3_{v_iw_i}, x^3_{v_iw_i}x^4_{v_iw_i}, x^4_{v_iw_i}w_i,\ w_ix^6_{v_iw_i}, x^6_{v_iw_i}s : i \in [k]\}$.

We define P by:
$P = \{(u, v) : u, v \in V\} \cup \{(x^5_{v_iw_i}, x^2_{v_iw_i}), (v_i, x^3_{v_iw_i}), (x^1_{v_iw_i}, x^4_{v_iw_i}), (x^2_{v_iw_i}, w_i), (x^3_{v_iw_i}, x^6_{v_iw_i}) : i \in [k]\}$.

Suppose there is a vertex coloring of G' with two colors such that all pairs $(u, v) \in P$ are rainbow vertex-connected. Observe that G is a subgraph of G'. For all $(u, v) \in V \times V$, they are belong to P and the paths connect them with length no more than 3 are belong to G, thus G is rainbow vertex-connected. We have $c(v_i) \neq c(w_i)$, since $\{(x^5_{v_iw_i}, x^2_{v_iw_i}), (v_i, x^3_{v_iw_i}), (x^1_{v_iw_i}, x^4_{v_iw_i}), (x^2_{v_iw_i}, w_i), (x^3_{v_iw_i}, x^6_{v_iw_i}) : i \in [k]\}$ are rainbow vertex-connected in G'.

On the other hand, if there is a 2-vertex coloring c of G such that G is rainbow vertex-connected and v_i, w_i are colored differently, we color G' with coloring c' as follows. For $v \in V$, $c'(v) = c(v)$. If $c(v_i) = 1,\ c(w_i) = 2$, then $c'(x^1_{v_iw_i}) = c'(x^3_{v_iw_i}) = 2$, $c'(x^2_{v_iw_i}) = c'(x^4_{v_iw_i}) = 1$. If $c(v_i) = 2,\ c(w_i) = 1$, then $c'(x^1_{v_iw_i}) = c'(x^3_{v_iw_i}) = 1$, $c'(x^2_{v_iw_i}) = c'(x^4_{v_iw_i}) = 2$. For all other vertices, we assign them by color 1 or 2 arbitrarily. It is easy to check that all $(u, v) \in P$ are rainbow vertex-connected. ∎

Lemma 5 *3-SAT $\preceq$ Problem 5.*

Proof. Let ϕ be an instance of 3-SAT with clauses $c_1, c_2, \ldots, c_m$ and variables $x_1, x_2, \ldots, x_n$. We construct a new graph G_ϕ and define two disjoint vertex sets with a one to one corresponding f. Add two new vertices s, t. Set

$$V_\phi = \{c_i : i \in [m]\} \cup \{x_i, \overline{x}_i : i \in [n]\} \cup \{s, t\}$$

and
$E_\phi = \{c_ic_j : i, j \in [m]\} \cup \{tx_i, t\overline{x}_i : i \in [n]\} \cup \{x_ic_j : x_i \in c_j\} \cup \{\overline{x}_ic_j : \overline{x}_i \in c_j\} \cup \{st\}$.

We define $V_1 = \{x_1, x_2, \ldots, x_n\}$, $V_2 = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\}$ and $f : V_1 \to V_2$ satisfying that $f(x_i) = \overline{x}_i$. Now we show that G_ϕ is 2-rainbow vertex-connected with different colors between x_i and $\overline{x}_i$ if and only if ϕ is satisfiable.

Suppose there is a vertex coloring $c : V_\phi \to \{0, 1\}$ such that G_ϕ is rainbow vertex-connected and x_i, $\overline{x}_i$ are colored differently. We first suppose $c(t) = 0$, set the value of x_i as the corresponding color of x_i. For each i, consider the rainbow vertex-connected path Q between vertices s and c_i, there must exist some j such that we can write $Q = stx_jc_i$ or $Q = st\overline{x}_jc_i$. Without loss of generality, suppose $Q = stx_jc_i$. Since $c(t) = 0$, we have $c(x_j) = 1$. Thus, the value of x_j is 1, which concludes that $c_i = 1$ as $x_j \in c_i$ by the construction of G_ϕ. For the other case $c(t) = 1$, we set $x_i = 1$ if $c(x_i) = 0$ and $x_i = 0$ otherwise. By some similar discussions, we also have $\phi = 1$.

On the other hand, for a given truth assignment of ϕ, we color G_ϕ as follows: $c(t) = 0$ and $c(c_i) = 1$ for $i \in [m]$; if $x_i = 1$, then $c(x_i) = 1$ and $c(\overline{x}_i) = 0$; otherwise, $c(x_i) = 0$ and $c(\overline{x}_i) = 1$. We can easily check that G_ϕ is rainbow vertex-connected. ∎

From the above three lemmas, we conclude our second theorem.

Theorem 2 *Given a graph G, deciding whether $rvc(G) = 2$ is NP-Complete. Thus, computing $rvc(G)$ is NP-Hard.*

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